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Global attractivity of a rational difference equation of order twenty

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ABSTRACT

In this research, qualitative behavior and periodic nature of the solutions of the difference equation $z_{n+1} = \alpha z_{n-9} + \frac{\beta z_{n-9}^2}{\gamma z_{n-9}+\delta z_{n-19}}$, $n = 0,1,2,\ldots$ has been studied where the initial conditions $z_{-19}, z_{-18}, \ldots, z_0$ are arbitrary positive real numbers and $\alpha, \beta, \gamma, \delta$ are constants. Solutions of some special cases of considered equation have been obtained.

1. Introduction

This paper deals with the solution behavior of the difference equation:

$$z_{n+1} = \alpha z_{n-9} + \frac{\beta z_{n-9}^2}{\gamma z_{n-9}+\delta z_{n-19}}, \qquad n = 0,1,2,\ldots \tag{1}$$

with initial conditions $z_{-19}, \ldots, z_0$ are arbitrary positive real numbers and $\alpha, \beta, \gamma, \delta$ are constants. We obtain solutions of some special cases of this equation. Difference equation is a vast field which impact almost found in every branch of pure as well as applied mathematics. Recently great interest is developed in studying difference equation systems. The reason is that there is need of some techniques whose can be used in investigating problems in various fields. Recently a great work is being done in studying the qualitative analysis of rational difference equations. Difference equations are very simple in form but it is very difficult to understand the behavior of their solutions (Ahmed and Youssef, 2013; Alghamdi et al., 2013; Asiri et al., 2015; Das and Bayram, 2010; Din, 2015).

Khaliq and Elsayed (2016) studied qualitative properties of difference equation of order six, $x_{n+1} = \alpha x_{n-2} + \frac{\beta x_{n-2}^2}{\gamma x_{n-2}+\delta x_{n-5}}$. Khaliq et al. (2016) studied global attractively of difference equations of order ten, $x_{n+1} = a x_{n-4} + \frac{b x_{n-4}^2}{c x_{n-4}+d x_{n-9}}$. Elabbasy et al. (2012) studied behavior of solutions of difference equations of order four, $x_{n+1} = a x_{n-4} + \frac{b x_{n-1}^2}{c x_{n-1}+d x_{n-3}}$. Elsayed and El-Dessoky (2013) examined the dynamics and global behavior of rational difference equation of order four, $x_{n+1} = a x_n + \frac{b x_n x_{n-2}}{c x_{n-2}+d x_{n-3}}$. El-Moneam and Alamoudy (2014) studied the positive solutions of the difference equation, $x_{n+1} = a x_n + \frac{b x_{n-1}+c x_{n-2}+f x_{n-3}+r x_{n-4}}{d x_{n-1}+e x_{n-2}+g x_{n-3}+s x_{n-4}}$. Elsayed (2011) investigated the solutions of following non-linear difference equation, $x_{n+1} = a x_{n-1} + \frac{b x_n x_{n-1}}{c x_n+d x_{n-2}}$. Karatas et al. (2006) gave solutions of the following difference equation, $x_{n+1} = \frac{x_{n-5}}{1+x_{n-2}x_{n-5}}$. Saleh and Aloqeili (2006) studied the solution of difference equation $y_{n+1} = A + \frac{y_n}{y_{n-k}}$. Yalcinkaya (2009) studied the boundedness, global stability, periodic behavior of difference equation and obtained its solutions, $x_{n+1} = \frac{a x_{n-k}}{b+c x_n^p}$. Yalcinkaya and Cinar (2009) has explored the difference equation $x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}$. For other relevant work on difference equations see (Touafek and Haddad, 2015; Yazlik et al., 2014, 2015; Zayed, 2014; Zhang et al., 2014). Suppose that I is some interval of real numbers and F a continuous function defined on $I^{k+1}(k+1 copies of I)$, where k is some natural number. We consider the following difference equation:

$$z_{n+1} = f(z_n, z_{n-1}, \ldots, z_{n-k}), \qquad n = 0,1,2,\ldots. \tag{2}$$

For given initial values $z_{-k}, z_{-k+1}, \ldots, z_0 \in I$ .The difference equation has a unique solution $\{z_n\}_{n=-19}^{\infty}$.

Definition: (Equilibrium point) A point $\bar{z}$ is an equilibrium point of Eq. 2 if

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$$\overline{z} = f(\overline{z}, \overline{z}, \dots), \qquad (3)$$

then, $z_n = \overline{z}$ for $n \geq 0$ is a solution of Eq. 2.

Definition: (Periodicity) A solution $\{z_n\}_{n=-k}^{\infty}$ of Eq. 2 is called periodic with period p if there exists an integer $p \geq 1$ such that $z_{n+p} = z_n$ for all $n \geq -k$.

Definition: (Fibonacci sequence) The $\{F_m\}_{m=1}^{\infty} = \{1,2,3,5,8,\dots\}$ that is $F_m = F_{m-1} + F_{m-2} \geq 0$ with initial conditions $F_{-2} = 0, F_{-1} = 1$ is Fibonacci sequence.

Definition: (Stability)

- If for every $\rho > 0$ there exist $\eta > 0$ such that for all with $z_{-k}, z_{-k+1}, \dots, z_0 \in I$ with $\sum_{\alpha=-k}^{0} |z_\alpha - \overline{z}| < \eta$, we have $|z_\alpha - \overline{z}| < \rho$ for all $n \geq -k$.Then equilibrium point $\overline{z}$ of difference Eq. 2 is called locally stable.
- If equilibrium point $\overline{z}$ is locally stable, and there exist $\beta > 0$ such for all initial values $z_{-k}, z_{-k+1}, \dots, z_0 \in I$ with $\sum_{\alpha=-k}^{0} |z_\alpha - \overline{z}| < \beta$, we have, $lim_{n\to\infty} z_n = \overline{z}$.Then $\overline{z}$ of difference Eq. 2 is called locally asymptotically stable.
- If $z_{-k}, z_{-k+1}, \dots, z_0 \in I$ always implies that $lim_{n\to\infty} z_n = \overline{z}$. Then $\overline{z}$ of Eq. 2 is called global attractor.
- If $\overline{z}$ is locally asymptotically stable as well as an attractor. Then equilibrium point $\overline{z}$ of difference Eq. 2 is called global asymptotically stable.
- The equilibrium point $\overline{z}$ of Eq. 2 is called unstable if it is not locally stable.
- The linearized Eq. 2 about the equilibrium point $\overline{z}$ is the linear difference equation:

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\overline{z}, \overline{z}, \dots \overline{z})}{\partial z_{n-i}} . y_{n-i}$$

1.1. Theorem A

Assume that $p_i \in R$ and $k \in (0,1,2,\dots)$. Then $\sum_{i=1}^{k} |p_i| < 1$ is a sufficient condition for the asymptotic stability of the difference equation

$$z_{n+k} + p_1 z_{n+k-1} + \dots + p_k z_n = 0 \qquad n = 0,1,\dots$$

1.2. Theorem B

Let $[\alpha, \beta]$ be a real numbers interval and suppose that $g: [\alpha, \beta]^2 \to [\alpha, \beta]$ is a continuous function and consider the equation

$$z_{n+1} = g(z_n, z_{n-1}), \qquad n = 0,1,\dots \qquad (4)$$

with the following two conditions:

(a) $g(x, y)$ is increasing function in $x \in [\alpha, \beta]$ for each fixed $y \in [\alpha, \beta]$ and $g(x, y)$ is decreasing in $y \in [\alpha, \beta]$ for each fixed $x \in [\alpha, \beta]$.
(b) If $(w, W) \in [\alpha, \beta] \times [\alpha, \beta]$ is a solution of the system $W = g(W, w)$ and $w = g(w, W)$ then $W = w$

Then Eq. 4 has a unique equilibrium point $\overline{z} \in [\alpha, \beta]$ and every solution of Eq. 4 converges to $\overline{z}$.

2. Local stability of equilibrium point of Eq. 1

The equilibrium point of Eq. 1 is given by

$$\overline{z} = \alpha\overline{z} + \frac{\beta\overline{z}^2}{\gamma\overline{z} + \delta\overline{z}}$$
$$\overline{z}^2(1-\alpha)(\gamma+\delta) = \beta\overline{z}^2$$

If $(1-\alpha)(\gamma+\delta) \neq \beta$ then the unique equilibrium point is $\overline{z} = 0$. Let $f: (0,\infty) \times (0,\infty) \to (0,\infty)$ be continuous and differentiable function defined as

$$f(u, v) = \alpha u + \frac{\beta u^2}{\gamma u + \delta v} \qquad (5)$$
$$\frac{\partial f(\overline{z}, \overline{z})}{\partial u} = \alpha + \frac{\beta\gamma + 2\beta\delta}{(\gamma+\delta)^2}$$
$$\frac{\partial f(\overline{z}, \overline{z})}{\partial v} = -\frac{\beta\delta}{(\gamma+\delta)^2}$$

The linearized equation of Eq. 1 about equilibrium point $\overline{z}$ is

$$y_{n+1} - [\alpha + \frac{\beta\gamma + 2\beta\delta}{(\gamma+\delta)^2}] y_n + [\frac{\beta\delta}{(\gamma+\delta)^2}] y_{n-1} = 0$$

2.1. Theorem

Assume that $\beta(\gamma + 3\delta) < (\gamma+\delta)^2(1-\alpha), \alpha < 1$. Then the equilibrium point $\overline{z} = 0$ of Eq. 1 is locally asymptotically stable.

Proof: Eq. 1 is asymptotically stable if

$$|\alpha + \frac{\beta\gamma + 2\beta\delta}{(\gamma+\delta)^2}| + |-\frac{\beta\delta}{(\gamma+\delta)^2}| < 1$$
$$\beta(\gamma + 3\delta) < (\gamma+\delta)^2(1-\alpha)$$

3. Global attractivity of equilibrium point of Eq. 1

3.1. Theorem

The equilibrium point $\overline{z} = 0$ of Eq. 1 is global attractor if $\gamma(1-\alpha) \neq \beta$.

Proof: Let α, β are real numbers and suppose that $g: [\alpha, \beta]^2 \to [\alpha, \beta]$ be function defined by Eq. 5. Suppose that (w, W) is a solution.

$$W = g(W, w) \quad and \quad w = g(w, W)$$

from Eq. 1, we see that

$$W = \alpha W + \frac{\beta W^2}{\gamma W + \delta w}, \qquad w = \alpha w + \frac{\beta w^2}{\gamma w + \delta W}$$
$$W(1-\alpha) = \frac{\beta W^2}{\gamma W + \delta w}, \qquad w(1-\alpha) = \frac{\beta w^2}{\gamma w + \delta W}$$

subtracting both above equations

$$\gamma(1-\alpha)(W^2 - w^2) = \beta(W^2 - w^2)$$

If $\gamma(1-\alpha) \neq \beta$ thus $w = W$. It concluded by theorem (B) that $\overline{z}$ is a global attractor of Eq. 1.

4. Bounded behavior of solutions of Eq. 1

4.1. Theorem

If $(\alpha+\frac{\beta}{\gamma})<1$, then every solution of Eq. 1 is bounded.

Proof: Let $\{z_n\}_{n=-19}^{\infty}$ be a solution of Eq. 1. Then

$$z_{n+1}=\alpha z_{n-9}+\frac{\beta z_{n-9}^2}{\gamma z_{n-9}+\delta z_{n-19}}\leq\alpha z_{n-9}+\frac{\beta z_{n-9}^2}{\gamma z_{n-9}}$$
$$=(\alpha+\frac{\beta}{\gamma})z_{n-9},$$

thus $z_{n+1}\leq z_{n-9}$ for all $n\geq 0$. Then the sub sequences:

$\{z_{10n-9}\}_{n=0}^{\infty},\{z_{10n-8}\}_{n=0}^{\infty},\{z_{10n-7}\}_{n=0}^{\infty},\{z_{10n-6}\}_{n=0}^{\infty},$
$\{z_{10n-5}\}_{n=0}^{\infty},\{z_{10n-4}\}_{n=0}^{\infty},\{z_{10n-3}\}_{n=0}^{\infty},\{z_{10n-2}\}_{n=0}^{\infty},$

$\{z_{10n-1}\}_{n=0}^{\infty}$ and $\{z_{10n}\}_{n=0}^{\infty}$ are decreasing and bounded from above by

$$M=\max\{z_{-19},z_{-18},z_{-17},z_{-16},z_{-15},z_{-14},z_{-13},z_{-12},z_{-11},z_{-10},z_{-9},z_{-8},z_{-7},z_{-6},z_{-5},z_{-4},z_{-3},z_{-2},z_{-1},z_0\}$$

Numerical examples: To confirm the result we take some numerical examples.

$z_{-19}=2,z_{-18}=5,z_{-17}=9,z_{-16}=8,z_{-15}=7,z_{-14}=5,z_{-13}=0,z_{-12}=11,z_{-11}=20,z_{-10}=9,z_{-9}=1,z_{-8}=6,z_{-7}=11,z_{-6}=10,z_{-5}=5,z_{-4}=8,z_{-3}=2,z_{-2}=9,z_{-1}=5,z_0=15,\alpha=0.6,\beta=2,\gamma=5,\delta=8$ (Fig. 1);

and if we take

$z_{-19}=3,z_{-18}=9,z_{-17}=15,z_{-16}=17,z_{-15}=11,z_{-14}=18,z_{-13}=0,z_{-12}=20,z_{-11}=13,z_{-10}=19,z_{-9}=5,z_{-8}=7,z_{-7}=8,z_{-6}=10,z_{-5}=14,z_{-4}=7,z_{-3}=10,z_{-2}=3,z_{-1}=5,z_0=2,\alpha=0.5,\beta=3,\gamma=11,\delta=15$ (Fig. 2).

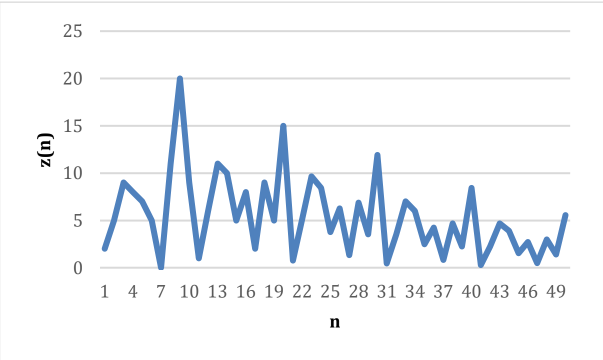

Fig. 1: Behavior of $z_{n+1}=\alpha z_{n-9}+\frac{\beta z_{n-9}^2}{\gamma z_{n-9}+\delta z_{n-19}}$

5. Different cases of Eq. 1

5.1. First equation

We study the special case of Eq. 1:

$$z_{n+1}=z_{n-9}+\frac{z_{n-9}^2}{z_{n-9}+z_{n-19}},\quad (6)$$

where the initial conditions

$z_{-19},z_{-18},z_{-17},z_{-16},z_{-15},z_{-14},z_{-13},z_{-12},z_{-11},z_{-10},z_{-9},z_{-8},z_{-7},z_{-6},z_{-5},z_{-4},z_{-3},z_{-2},z_{-1},z_0$

are positive arbitrary real numbers.

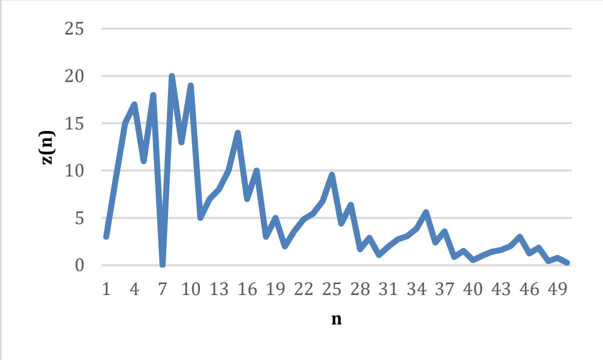

Fig. 2: Behavior of $z_{n+1}=\alpha z_{n-9}+\frac{\beta z_{n-9}^2}{\gamma z_{n-9}+\delta z_{n-19}}$

5.2. Theorem

Let $\{z_n\}_{n=-19}^{\infty}$ be solution of Eq. 6. Then for

$z_{10n-9}=w\prod_{i=1}^{n}[\frac{F_{2i+1}k+F_{2i}w}{F_{2i}k+F_{2i-1}w}]$, $z_{10n-8}=j\prod_{i=1}^{n}[\frac{F_{2i+1}j+F_{2i}t}{F_{2i}j+F_{2i-1}t}]$
$z_{10n-7}=h\prod_{i=1}^{n}[\frac{F_{2i+1}h+F_{2i}s}{F_{2i}h+F_{2i-1}s}]$, $z_{10n-6}=g\prod_{i=1}^{n}[\frac{F_{2i+1}g+F_{2i}r}{F_{2i}g+F_{2i-1}r}]$
$z_{10n-5}=f\prod_{i=1}^{n}[\frac{F_{2i+1}f+F_{2i}q}{F_{2i}f+F_{2i-1}q}]$, $z_{10n-4}=e\prod_{i=1}^{n}[\frac{F_{2i+1}e+F_{2i}p}{F_{2i}e+F_{2i-1}p}]$
$z_{10n-3}=d\prod_{i=1}^{n}[\frac{F_{2i+1}d+F_{2i}o}{F_{2i}d+F_{2i-1}o}]$, $z_{10n-2}=c\prod_{i=1}^{n}[\frac{F_{2i+1}c+F_{2i}n}{F_{2i}c+F_{2i-1}n}]$
$z_{10n-1}=b\prod_{i=1}^{n}[\frac{F_{2i+1}b+F_{2i}m}{F_{2i}b+F_{2i-1}m}]$, $z_{10n}=a\prod_{i=1}^{n}[\frac{F_{2i+1}a+F_{2i}l}{F_{2i}a+F_{2i-1}l}]$

where

$z_{-19}=w,z_{-18}=t,z_{-17}=s,z_{-16}=r,z_{-15}=q,z_{-14}=p,z_{-13}=o,z_{-12}=n,z_{-11}=m,z_{-10}=l,z_{-9}=k,z_{-8}=j,z_{-7}=h,z_{-6}=g,z_{-5}=f,z_{-4}=e,z_{-3}=d,z_{-2}=c,z_{-1}=b,z_0=a$

and

$[F_m]_{m=1}^{\infty}=1,2,3,5,8,\ldots$

Proof: We prove by mathematical induction the solutions of Eq. 6. First for $n=0$, the result holds. Assume the above results are satisfied for $n-1,n-2$.

$z_{10n-19}=w\prod_{i=1}^{n-1}[\frac{F_{2i+1}k+F_{2i}w}{F_{2i}k+F_{2i-1}w}]$, $z_{10n-18}=j\prod_{i=1}^{n-1}[\frac{F_{2i+1}j+F_{2i}t}{F_{2i}j+F_{2i-1}t}]$
$z_{10n-17}=h\prod_{i=1}^{n-1}[\frac{F_{2i+1}h+F_{2i}s}{F_{2i}h+F_{2i-1}s}]$, $z_{10n-16}=g\prod_{i=1}^{n-1}[\frac{F_{2i+1}g+F_{2i}r}{F_{2i}g+F_{2i-1}r}]$

$z_{10n-15} = f\prod_{i=1}^{n-1}\left[\frac{F_{2i+1}f+F_{2i}q}{F_{2i}f+F_{2i-1}q}\right]$, $z_{10n-14} = e\prod_{i=1}^{n-1}\left[\frac{F_{2i+1}e+F_{2i}p}{F_{2i}e+F_{2i-1}p}\right]$

$z_{10n-13} = d\prod_{i=1}^{n-1}\left[\frac{F_{2i+1}d+F_{2i}o}{F_{2i}d+F_{2i-1}o}\right]$, $z_{10n-12} = c\prod_{i=1}^{n-1}\left[\frac{F_{2i+1}c+F_{2i}n}{F_{2i}c+F_{2i-1}n}\right]$

$z_{10n-11} = b\prod_{i=1}^{n-1}\left[\frac{F_{2i+1}b+F_{2i}m}{F_{2i}b+F_{2i-1}m}\right]$, $z_{10n-10} = a\prod_{i=1}^{n-1}\left[\frac{F_{2i+1}a+F_{2i}l}{F_{2i}a+F_{2i-1}l}\right]$

$z_{10n-29} = w\prod_{i=1}^{n-2}\left[\frac{F_{2i+1}k+F_{2i}w}{F_{2i}k+F_{2i-1}w}\right]$, $z_{10n-28} = j\prod_{i=1}^{n-2}\left[\frac{F_{2i+1}j+F_{2i}t}{F_{2i}j+F_{2i-1}t}\right]$

$z_{10n-27} = h\prod_{i=1}^{n-2}\left[\frac{F_{2i+1}h+F_{2i}s}{F_{2i}h+F_{2i-1}s}\right]$, $z_{10n-26} = g\prod_{i=1}^{n-2}\left[\frac{F_{2i+1}g+F_{2i}r}{F_{2i}g+F_{2i-1}r}\right]$

$z_{10n-25} = f\prod_{i=1}^{n-2}\left[\frac{F_{2i+1}f+F_{2i}q}{F_{2i}f+F_{2i-1}q}\right]$, $z_{10n-24} = e\prod_{i=1}^{n-2}\left[\frac{F_{2i+1}e+F_{2i}p}{F_{2i}e+F_{2i-1}p}\right]$

$z_{10n-23} = d\prod_{i=1}^{n-2}\left[\frac{F_{2i+1}d+F_{2i}o}{F_{2i}d+F_{2i-1}o}\right]$, $z_{10n-22} = c\prod_{i=1}^{n-2}\left[\frac{F_{2i+1}c+F_{2i}n}{F_{2i}c+F_{2i-1}n}\right]$

$z_{10n-21} = b\prod_{i=1}^{n-2}\left[\frac{F_{2i+1}b+F_{2i}m}{F_{2i}b+F_{2i-1}m}\right]$, $z_{10n-20} = a\prod_{i=1}^{n-2}\left[\frac{F_{2i+1}a+F_{2i}l}{F_{2i}a+F_{2i-1}l}\right]$

from Eq. 6

$$z_{10n-1} = z_{10n-11} + \frac{z_{10n-11}^2}{z_{10n-11}+z_{10n-21}}$$

$$= b\prod_{i=1}^{a-1}\left[\frac{F_{2i+1}b+F_{2i}m}{F_{2i}b+F_{2i-1}m}\right] + \frac{b^2\prod_{i=1}^{n-1}\left[\frac{F_{2i+1}b+F_{2i}m}{F_{2i}b+F_{2i-1}m}\right]\left[\left(\frac{F_3b+F_2m}{F_2b+F_1m}\right);\dots;\left(\frac{F_{2n-3}b+F_{2n-4}m}{F_{2n-1}b+F_{2n-3}m}\right)\left(\frac{F_{2n-1}b+F_{2n-2}m}{F_{2n-2}b+F_{2n-3}m}\right)\right]}{b\prod_{i=1}^{n-1}\left[\frac{F_{2i+1}b+F_{2i}m}{F_{2i}b+F_{2i-1}m}\right]+b\prod_{i=1}^{n-1}\left[\frac{F_{2i+1}b+F_{2i}m}{F_{2i}b+F_{2i-1}m}\right]}$$

$$= b\prod_{i=1}^{n-1}\left[\frac{F_{2i+1}b+F_{2i}m}{F_{2i}b+F_{2i-1}m}\right] + \frac{b\prod_{i=1}^{n-1}\left[\frac{F_{2i+1}b+F_{2i}m}{F_{2i}b+F_{2i-1}m}\right]\left(\frac{F_{2n-1}b+F_{2n-2}m}{F_{2n-2}b+F_{2n-3}m}\right)}{\left(\frac{F_{2n-1}b+F_{2n-2}m}{F_{2n-2}b+F_{2n-3}m}+1\right)}$$

$$= b\prod_{i=1}^{n-1}\left[\frac{F_{2i+1}b+F_{2i}m}{F_{2i}b+F_{2i-1}m}\right] + \frac{b\prod_{i=1}^{n-1}\left[\frac{F_{2i+1}b+F_{2i}m}{F_{2i}b+F_{2i-1}m}\right](F_{2n-1}b+F_{2n-2}m)}{(F_{2n-2}b+F_{2n-3}m+F_{2n-2}b+F_{2n-3}m)}$$

$$= b\prod_{i=1}^{n-1}\left[\frac{F_{2i+1}b+F_{2i}m}{F_{2i}b+F_{2i-1}m}\right] + \frac{b\prod_{i=1}^{n-1}\left[\frac{F_{2i+1}b+F_{2i}m}{F_{2i}b+F_{2i-1}m}\right](F_{2n-1}b+F_{2n-2}m)}{(F_{2n}b+F_{2n-1}m)}$$

$$= b\prod_{i=1}^{n-1}\left[\frac{F_{2i+1}b+F_{2i}m}{F_{2i}b+F_{2i-1}m}\right]\left[1+\frac{(F_{2n-1}b+F_{2n-2}m)}{(F_{2n}b+F_{2n-1}m)}\right]$$

$$= b\prod_{i=1}^{n-1}\left[\frac{F_{2i+1}b+F_{2i}m}{F_{2i}b+F_{2i-1}m}\right]\left[\frac{F_{2n+1}b+F_{2n}m}{F_{2n}b+F_{2n-1}m}\right]$$

$$= b\prod_{i=1}^{n}\left[\frac{F_{2i+1}b+F_{2i}m}{F_{2i}b+F_{2i-1}m}\right]$$

similarly

$$z_{10n} = z_{10n-10} + \frac{z_{10n-10}^2}{z_{10n-10}+z_{10n-20}}$$

$$= a\prod_{i=1}^{a-1}\left[\frac{F_{2i+1}a+F_{2i}l}{F_{2i}a+F_{2i-1}l}\right] + \frac{a^2\prod_{i=1}^{n-1}\left[\frac{F_{2i+1}a+F_{2i}l}{F_{2i}a+F_{2i-1}l}\right]\left[\left(\frac{F_3a+F_2l}{F_2a+F_1l}\right);\dots;\left(\frac{F_{2n-3}a+F_{2n-4}l}{F_{2n-1}a+F_{2n-3}l}\right)\left(\frac{F_{2n-1}a+F_{2n-2}l}{F_{2n-2}a+F_{2n-3}l}\right)\right]}{a\prod_{i=1}^{n-1}\left[\frac{F_{2i+1}a+F_{2i}l}{F_{2i}a+F_{2i-1}l}\right]+a\prod_{i=1}^{n-1}\left[\frac{F_{2i+1}a+F_{2i}l}{F_{2i}a+F_{2i-1}l}\right]}$$

$$= a\prod_{i=1}^{n-1}\left[\frac{F_{2i+1}a+F_{2i}l}{F_{2i}a+F_{2i-1}l}\right] + \frac{a\prod_{i=1}^{n-1}\left[\frac{F_{2i+1}a+F_{2i}l}{F_{2i}a+F_{2i-1}l}\right]\left(\frac{F_{2n-1}a+F_{2n-2}l}{F_{2n-2}a+F_{2n-3}l}\right)}{\left(\frac{F_{2n-1}a+F_{2n-2}l}{F_{2n-2}a+F_{2n-3}l}+1\right)}$$

$$= a\prod_{i=1}^{n-1}\left[\frac{F_{2i+1}a+F_{2i}l}{F_{2i}a+F_{2i-1}l}\right] + \frac{a\prod_{i=1}^{n-1}\left[\frac{F_{2i+1}a+F_{2i}l}{F_{2i}a+F_{2i-1}l}\right](F_{2n-1}a+F_{2n-2}l)}{(F_{2n-2}a+F_{2n-3}l+F_{2n-2}a+F_{2n-3}l)}$$

$$= a\prod_{i=1}^{n-1}\left[\frac{F_{2i+1}a+F_{2i}l}{F_{2i}a+F_{2i-1}l}\right] + \frac{a\prod_{i=1}^{n-1}\left[\frac{F_{2i+1}a+F_{2i}l}{F_{2i}a+F_{2i-1}l}\right](F_{2n-1}a+F_{2n-2}l)}{(F_{2n}a+F_{2n-1}l)}$$

$$= a\prod_{i=1}^{n-1}\left[\frac{F_{2i+1}a+F_{2i}l}{F_{2i}a+F_{2i-1}l}\right]\left[1+\frac{(F_{2n-1}a+F_{2n-2}l)}{(F_{2n}a+F_{2n-1}l)}\right]$$

$$= a\prod_{i=1}^{n-1}\left[\frac{F_{2i+1}a+F_{2i}l}{F_{2i}a+F_{2i-1}l}\right]\left[\frac{F_{2n+1}a+F_{2n}l}{F_{2n}a+F_{2n-1}l}\right]$$

$$= a\prod_{i=1}^{n}\left[\frac{F_{2i+1}a+F_{2i}l}{F_{2i}a+F_{2i-1}l}\right].$$

Numerical example: We now confirm result by taking numerical example for

$z_{-19} = 10, z_{-18} = 5, z_{-17} = 8, z_{-16} = 9, z_{-15} = 2, z_{-14} = 7, z_{-13} = 1, z_{-12} = 5, z_{-11} = 11, z_{-10} = 2, z_{-9} = 3, z_{-8} = 10, z_{-7} = 8, z_{-6} = 15, z_{-5} = 2, z_{-4} = 10, z_{-3} = 7, z_{-2} = 5, z_{-1} = 1, z_0 = 8$ (Fig. 3)

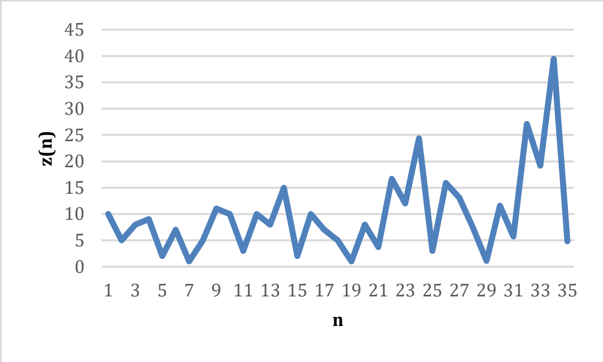

Fig. 3: Behavior of $z_{n+1} = z_{n-9} + \frac{z_{n-9}^2}{z_{n-9}+z_{n-19}}$

5.3. Second equation

We take form of Eq. 1

$$z_{n+1} = z_{n-9} + \frac{z_{n-9}^2}{z_{n-9}-z_{n-19}}, \quad (7)$$

where the initial conditions

$z_{-19}, z_{-18}, z_{-17}, z_{-16}, z_{-15}, z_{-14}, z_{-13}, z_{-12}, z_{-11}, z_{-10}, z_{-9}, z_{-8}, z_{-7}, z_{-6}, z_{-5}, z_{-4}, z_{-3}, z_{-2}, z_{-1}, z_0$

are positive arbitrary real numbers.

Theorem: Let $\{z_n\}_{n=-19}^{\infty}$ be a solution of Eq. 7. Then for $n = 0,1,2,..$

$z_{10n-9} = w\prod_{i=1}^{n}\left[\frac{F_{i+2}k-F_iw}{F_ik-F_{i-2}w}\right]$, $z_{10n-8} = j\prod_{i=1}^{n}\left[\frac{F_{i+2}j-F_it}{F_ij-F_{i-2}t}\right]$

$z_{10n-7} = h\prod_{i=1}^{n}\left[\frac{F_{i+2}h-F_is}{F_ih-F_{i-2}s}\right]$, $z_{10n-6} = g\prod_{i=1}^{n}\left[\frac{F_{i+2}g-F_ir}{F_ig-F_{i-2}r}\right]$

$z_{10n-5} = f\prod_{i=1}^{n}\left[\frac{F_{i+2}f-F_iq}{F_if-F_{i-2}q}\right]$, $z_{10n-4} = e\prod_{i=1}^{n}\left[\frac{F_{i+2}e-F_ip}{F_ie-F_{i-2}p}\right]$

$z_{10n-3} = d\prod_{i=1}^{n}\left[\frac{F_{i+2}d-F_io}{F_id-F_{i-2}o}\right]$, $z_{10n-2} = c\prod_{i=1}^{n}\left[\frac{F_{i+2}c-F_in}{F_ic-F_{i-2}n}\right]$

$z_{10n-1} = b\prod_{i=1}^{n}\left[\frac{F_{i+2}b-F_im}{F_ib-F_{i-2}m}\right]$, $z_{10n} = a\prod_{i=1}^{n}\left[\frac{F_{i+2}a-F_il}{F_ia-F_{i-2}l}\right]$

where

$z_{-19} = w, z_{-18} = t, z_{-17} = s, z_{-16} = r, z_{-15} = q, z_{-14} = p, z_{-13} = o, z_{-12} = n, z_{-11} = m, z_{-10} = l, z_{-9} = k, z_{-8} = j, z_{-7} = h, z_{-6} = g, z_{-5} = f, z_{-4} = e, z_{-3} = d, z_{-2} = c, z_{-1} = b, z_0 = a$ and $[F_m]_{m=1}^{\infty} = 1,2,3,5,8,\dots$

Proof: Proof is same as previous and omitted.

Numerical example: We will take some numerical examples to confirm result.

$z_{-19} = 5, z_{-18} = 15, z_{-17} = 18, z_{-16} = 20, z_{-15} = 19, z_{-14} = 17, z_{-13} = 18, z_{-12} = 20, z_{-11} = 13, z_{-10} = 4, z_{-9} = 11, z_{-8} = 8, z_{-7} = 10, z_{-6} = 12, z_{-5} = 9, z_{-4} = 7, z_{-3} = 5, z_{-2} = 1, z_{-1} = 3, z_0 = 7$ (Fig. 4)

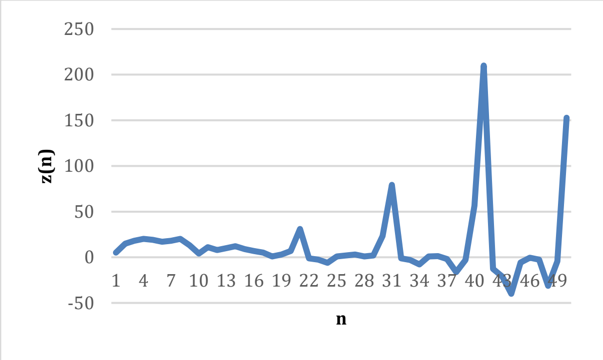

Fig. 4: Behavior of $z_{n+1} = z_{n-9} + \frac{z_{n-9}^2}{z_{n-9}-z_{n-19}}$

5.4. Third equation

We discuss the form of solutions of Eq. 1 in this section

$$z_{n+1} = z_{n-9} - \frac{z_{n-9}^2}{z_{n-9}+z_{n-19}}, \tag{8}$$

where the initial conditions

$z_{-19}, z_{-18}, z_{-17}, z_{-16}, z_{-15}, z_{-14}, z_{-13}, z_{-12}, z_{-11}, z_{-10}, z_{-9}, z_{-8}, z_{-7}, z_{-6}, z_{-5}, z_{-4}, z_{-3}, z_{-2}, z_{-1}, z_0$

are positive arbitrary real numbers.

Theorem: The solution of Eq. 5 will take the following formulas for $n = 0,1,2,..$

$$z_{10n-9} = \frac{kw}{F_n k + F_{n+1} w}, \quad z_{10n-8} = \frac{jt}{F_n j + F_{n+1} t}$$
$$z_{10n-7} = \frac{hs}{F_n h + F_{n+1} s}, \quad z_{10n-6} = \frac{gr}{F_n g + F_{n+1} r}$$
$$z_{10n-5} = \frac{fq}{F_n f + F_{n+1} q}, \quad z_{10n-4} = \frac{ep}{F_n e + F_{n+1} p}$$
$$z_{10n-3} = \frac{od}{F_n o + F_{n+1} d}, \quad z_{10n-2} = \frac{cn}{F_n c + F_{n+1} n}$$
$$z_{10n-1} = \frac{bm}{F_n b + F_{n+1} m}, \quad z_{10n} = \frac{al}{F_n a + F_{n+1} l}$$

Proof: For $n = 0$ the result is obvious.Suppose that $n > 0$ and the assumption holds for $n-1, n-2$.

$$z_{10n-19} = \frac{kw}{F_{n-1} k + F_n w}, \quad z_{10n-18} = \frac{jt}{F_{n-1} j + F_n t}$$
$$z_{10n-17} = \frac{hs}{F_{n-1} h + F_n s}, \quad z_{10n-16} = \frac{gr}{F_{n-1} g + F_n r}$$
$$z_{10n-15} = \frac{fq}{F_{n-1} f + F_n q}, \quad z_{10n-14} = \frac{ep}{F_{n-1} e + F_n p}$$
$$z_{10n-13} = \frac{od}{F_{n-1} o + F_n d}, \quad z_{10n-12} = \frac{cn}{F_{n-1} c + F_n n}$$
$$z_{10n-11} = \frac{bm}{F_{n-1} b + F_n m}, \quad z_{10n-10} = \frac{al}{F_{n-1} a + F_n l}$$
$$z_{10n-29} = \frac{kw}{F_{n-2} k + F_{n-1} w}, \quad z_{10n-28} = \frac{jt}{F_{n-2} j + F_{n-1} t}$$
$$z_{10n-27} = \frac{hs}{F_{n-2} h + F_{n-1} s}, \quad z_{10n-26} = \frac{gr}{F_{n-2} g + F_{n-1} r}$$
$$z_{10n-25} = \frac{fq}{F_{n-2} f + F_{n-1} q}, \quad z_{10n-24} = \frac{ep}{F_{n-2} e + F_{n-1} p}$$
$$z_{10n-23} = \frac{od}{F_{n-2} o + F_{n-1} d}, \quad z_{10n-22} = \frac{cn}{F_{n-2} c + F_{n-1} n}$$
$$z_{10n-21} = \frac{bm}{F_{n-2} b + F_{n-1} m}, \quad z_{10n-20} = \frac{al}{F_{n-2} a + F_{n-1} l}$$

now from Eq. 8

$$z_{10n-1} = z_{10n-11} - \frac{z_{10n-11}^2}{z_{10n-11}+z_{10n-21}}$$
$$= [\frac{bm}{F_{n-1}b+F_n m}] - \frac{[\frac{bm}{F_{n-1}b+F_n m}][\frac{bm}{F_{n-1}b+F_n m}]}{[\frac{bm}{F_{n-1}b+F_n m}]+[\frac{bm}{F_{n-2}b+F_{n-1}m}]}$$
$$= [\frac{bm}{F_{n-1}b+F_n m}] - [\frac{bm}{F_{n-1}b+F_n m} \times \frac{F_{n-2}b+F_{n-1}m}{F_{n-2}b+F_{n-1}m+F_{n-1}b+F_n m}]$$
$$= [\frac{bm}{F_{n-1}b+F_n m}] - [1 - \frac{F_{n-2}b+F_{n-1}m}{F_n b+F_{n+1}m}]$$
$$= \frac{bm}{F_n b+F_{n+1}m}$$

Similarly

$$z_{10n} = z_{10n-10} - \frac{z_{10n-10}^2}{z_{10n-10}+z_{10n-20}}$$
$$= [\frac{al}{F_{n-1}a+F_n l}] - \frac{[\frac{al}{F_{n-1}a+F_n l}][\frac{al}{F_{n-1}a+F_n l}]}{[\frac{al}{F_{n-1}a+F_n l}]+[\frac{al}{F_{n-2}a+F_{n-1}l}]}$$
$$= [\frac{al}{F_{n-1}a+F_n l}] - [\frac{al}{F_{n-1}a+F_n l} \times \frac{F_{n-2}a+F_{n-1}l}{F_{n-2}a+F_{n-1}l+F_{n-1}a+F_n l}]$$
$$= [\frac{al}{F_{n-1}a+F_n l}] - [1 - \frac{F_{n-2}a+F_{n-1}l}{F_n a+F_{n+1}l}]$$
$$= \frac{al}{F_n a+F_{n+1}l}$$

Numerical Example: We will take numerical example to confirm result.

$z_{-19} = 20, z_{-18} = 1, z_{-17} = 2, z_{-16} = 25, z_{-15} = 19, z_{-14} = 17, z_{-13} = 9, z_{-12} = 14, z_{-11} = 11, z_{-10} = 0, z_{-9} = 5, z_{-8} = 9, z_{-7} = 7, z_{-6} = 8, z_{-5} = 12, z_{-4} = 13, z_{-3} = 0, z_{-2} = 2, z_{-1} = 5, z_0 = 3$ (Fig. 5)

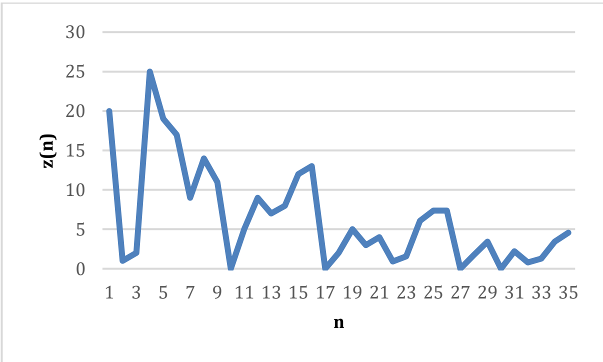

Fig. 5: Behavior of $z_{n+1} = z_{n-9} - \frac{z_{n-9}^2}{z_{n-9}+z_{n-19}}$

5.5. Fourth equation

In this part we discuss the solutions of form of Eq. 1:

$$z_{n+1} = z_{n-9} - \frac{z_{n-9}^2}{z_{n-9}-z_{n-19}}, \tag{9}$$

where the initial conditions

$z_{-19}, z_{-18}, z_{-17}, z_{-16}, z_{-15}, z_{-14}, z_{-13}, z_{-12}, z_{-11}, z_{-10}, z_{-9}, z_{-8}, z_{-7}, z_{-6}, z_{-5}, z_{-4}, z_{-3}, z_{-2}, z_{-1}, z_0$

are positive arbitrary real numbers.

5.5.1. Theorem: Assume that $\{z_n\}_{n=-19}^{\infty}$ be a solution of Eq. 9.Then every solution of Eq. 9 is periodic with period 60. Moreover $\{z_n\}_{n=-19}^{\infty}$ takes the form

$$\begin{bmatrix} w,t,s,r,q,p,o,n,m,l,k,j,h,g,f,e,d,c,b,a,\frac{-kw}{k-w},\frac{-ep}{e-p} \\ \frac{-jt}{j-t},\frac{-do}{d-o},\frac{-hs}{h-s},\frac{-cn}{c-n},\frac{-gr}{g-r} \\ \frac{-bm}{b-m},\frac{-fq}{f-q},\frac{-al}{a-l},-w,-t,-s,-r,-q,-p,-o,-n,-m,-l,-k,-j, \\ -h,-g,-f,-e,-d,-c,-b,-a,\frac{kw}{k-w},\frac{ep}{e-p},\frac{jt}{j-t},\frac{do}{d-o} \\ \frac{hs}{h-s},\frac{cn}{c-n},\frac{gr}{g-r},\frac{bm}{b-m},\frac{fq}{f-q} \\ \frac{al}{a-l},w,t,s,r,q,p,o,n,m,l,k,j,h,g,f,e,d,c,b,a,\ldots \end{bmatrix}$$

or

$z_{60n-19}=w,\quad z_{60n-18}=t,\quad z_{60n-17}=s$
$z_{60n-16}=r,\quad z_{60n-15}=q,\quad z_{60n-14}=p$
$z_{60n-13}=o,\quad z_{60n-12}=n,\quad z_{60n-11}=m$
$z_{60n-10}=l,\quad z_{60n-9}=k,\quad z_{60n-8}=j$
$z_{60n-7}=h,\quad z_{60n-6}=g,\quad z_{60n-5}=f$
$z_{60n-4}=e,\quad z_{60n-3}=d,\quad z_{60n-2}=c$
$z_{60n-1}=b,\quad z_{60n}=a,\quad z_{60n+1}=\frac{-kw}{k-w}$
$z_{60n+2}=\frac{-ep}{e-p},\quad z_{60n+3}=\frac{-jt}{j-t},\quad z_{60n+4}=\frac{-do}{d-o}$
$z_{60n+5}=\frac{-hs}{h-s},\quad z_{60n+6}=\frac{-cn}{c-n},\quad z_{60n+7}=\frac{-gr}{g-r}$
$z_{60n+8}=\frac{-bm}{b-m},\quad z_{60n+9}=\frac{-fq}{f-q},\quad z_{60n+10}=\frac{-al}{a-l}$
$z_{60n+11}=-w,\quad z_{60n+12}=-t,\quad z_{60n+13}=-s$
$z_{60n+14}=-r,\quad z_{60n+15}=-q,\quad z_{60n+16}=-p$
$z_{60n+17}=-o,\quad z_{60n+18}=-n,\quad z_{60n+19}=-m$
$z_{60n+20}=-l,\quad z_{60n+21}=-k,\quad z_{60n+22}=-j$
$z_{60n+23}=-h,\quad z_{60n+24}=-g,\quad z_{60n+25}=-f$
$z_{60n+26}=-e,\quad z_{60n+27}=-d,\quad z_{60n+28}=-c$
$z_{60n+29}=-b,\quad z_{60n+30}=-a,\quad z_{60n+31}=\frac{kw}{k-w}$
$z_{60n+32}=\frac{ep}{e-p},\quad z_{60n+33}=\frac{jt}{j-t},\quad z_{60n+34}=\frac{do}{d-o}$
$z_{60n+35}=\frac{hs}{h-s},\quad z_{60n+36}=\frac{cn}{c-n},\quad z_{60n+37}=\frac{gr}{g-r}$
$z_{60n+38}=\frac{bm}{b-m},\quad z_{60n+39}=\frac{fq}{f-q},\quad z_{60n+40}=\frac{al}{a-l}$

where

$z_{-19},z_{-18},z_{-17},z_{-16},z_{-15},z_{-14},z_{-13},z_{-12},z_{-11},z_{-10},$
$z_{-9},z_{-8},z_{-7},z_{-6},z_{-5},z_{-4},z_{-3},z_{-2},z_{-1},z_0$

Proof: Same proof as theorem Eq. 5 and will be omitted therefore.

Numerical Example: We will take numerical example to confirm result.

$z_{-19}=5, z_{-18}=9, z_{-17}=2, z_{-16}=3, z_{-15}=2, z_{-14}=1, z_{-13}=4, z_{-12}=6, z_{-11}=0, z_{-10}=10, z_{-9}=12, z_{-8}=15, z_{-7}=17, z_{-6}=18, z_{-5}=0, z_{-4}=10, z_{-3}=7, z_{-2}=8, z_{-1}=1, z_0=2$ (Fig. 6)

6. Conclusion

We studied the global stability, bounded behavior and forms of solutions of few cases of difference Eq. 1 and concluded that if $(1-\alpha)(\gamma+\delta)\neq\beta$ then the unique equilibrium point of Eq. 1 is $\overline{z}=0$. The equilibrium point $\overline{z}=0$ of Eq. 1 is locally asymptotically stable when $\beta(\gamma+3\delta)<(\gamma+\delta)^2(1-\alpha), \alpha<1$. The equilibrium point $\overline{z}=0$ of Eq. 1 is global attractor if $\gamma(1-\alpha)\neq\beta$. Every solution of Eq. 1 is bounded if $(\alpha+\frac{\beta}{\gamma})<1$. In the end we obtained solution of four different types of Eq. 1 and gave numerical examples of each case by assigning different initial values by using Matlab.

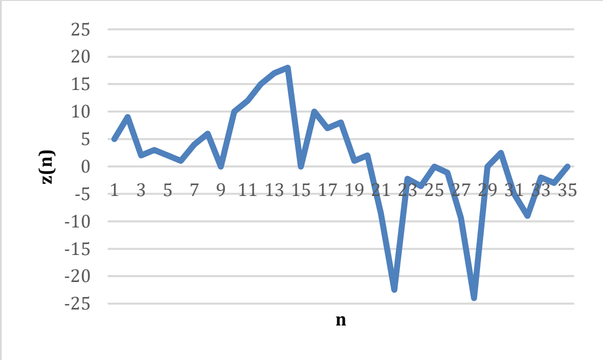

Fig. 6: Behavior of $z_{n+1}=z_{n-9}-\frac{z_{n-9}^2}{z_{n-9}-z_{n-19}}$

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